Periodic Solution of Plane Librational Motion of a Satellite

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This paper presents an approximate method of solution for the plane librational motion of an axially symmetric satellite in an elliptic orbit. Two types of motion are formulated, namely, small and finite oscillations. The equation of motion for small oscillation can be written in the form of the well-known Hill's equation with a sinusoidal forcing function. The periodic solution for finite oscillation is obtained in the form of a seven-term Fourier series in terms of the mean anomaly. The coefficients of the series are determined by using Lagrange's theorem of expansion of functions and Galerkin's principle. Numerical examples are presented for the case of $e = 0.2$ with $K = (I_t - I_a)/I_t$ as a parameter. The results show that for small and moderate eccentricity the series converges rapidly. Comparison of the series solutions with an exact solution is made.

1. Introduction

THE plane librational motion of an artificial satellite in an elliptic orbit has attracted great interest in recent years, because it has wide practical applications. The publications concerning this subject are numerous and only a few are listed in the references. The objectives of the present study are twofold. First, a direct formulation of the small oscillatory motion in an orbit with small eccentricity is presented. Second, a method of series solution for the periodic motion is developed. For mathematical simplicity, this analysis has been limited to the librational motion of a satellite in its orbital plane and assumes that the satellite has an axis of symmetry which lies toward the center of the attracting body. This simple model is generally acceptable for practical applications. The extension of the present development to the coupled in-plane and out-of-plane librational motions is under further study.

The plane librational motion of an axially symmetric satellite in an elliptic orbit can be derived in the form

$$I_t d^2/dt^2(f + \Psi) = -\tfrac{3}{2}(I_t - I_a)(\mu/r^3)\sin 2\Psi \quad (1)$$

where f is the true anomaly, r is the orbital radius, Ψ is the angle between the axis of symmetry of the satellite and the radius vector of the orbit, μ is the gravity constant, I_a and I_t are respectively, the axial and transverse moments of inertia of the satellite about its center of mass. It has been shown in Ref. 1 that the librational motion of a satellite has no significant effect on its orbital motion. If all the disturbing forces on the orbit are neglected and with the aid of the Keplerian orbital relationships

$$df/dt = h/r^2, \; dr/dt = (\mu e \sin f)/h \quad (2)$$

where h is the angular momentum. Equation (1) can be written

$$(1 + e\cos f)d^2\Psi/df^2 - 2e\sin f(d\Psi/df) + \tfrac{3}{2}K\sin 2\Psi = 2e\sin f \quad (3)$$

where $K = (I_t - I_a)/I_t$. The value of K is in the region -1–1. Since the region $-1 < K < 0$ has unstable librational motion, so the analysis given only considers $0 < K < 1$. Another form of Eq. (3) is readily obtained by using the mean anomaly M as the independent variable

$$d^2\Psi/dM^2 + \tfrac{3}{2}K(a/r)^3\sin 2\Psi = 2e(a/r)^3\sin f \quad (4)$$

where e and a are the eccentricity and semimajor axes of the orbit, respectively.

Moran[1] and Schechter[2] have treated Eq. (3) by a perturbation approach for an orbit with small eccentricity. Numerical investigation of the stability of the plane librational motion by using a phase space method has been made by Brereton and Modi.[3] Beletsky[4] has obtained a solution of Eq. (3) in series form for small eccentricity. Kill[5] has shown the existence of a periodic solution of Eq. (3) by means of Cesari's method.[6] Both an exact numerical solution and a series solution by the harmonic balance method have been published recently by Modi and Brereton[7] for Eq. (3). This paper presents a series solution of Eq. (4) in terms of M based on Lagrange's expansion theorem.

2. Analysis

To obtain the solution of Eq. (4) in terms of M, it is necessary to expand the functions $(a/r)^3$,$(a/r)^3 \sin f$, and $\sin f$ in Fourier series in the variable M. With the aid of the well-known series presented in Appendices A and B we may formulate the coefficients of these expansions

$$(a/r)^3 = \tfrac{1}{2}A_0 + \sum_{j=1} A_j \cos jM \quad (5)$$

$$(a/r)^3 \sin f = \sum_{j=1} B_j \sin jM \quad (6)$$

by the following two methods.

Method 1: Forming Product of Two Series

The Fourier coefficients can be expressed in terms of Bessel functions of the first kind $J_n(ne)$ by forming the product of the two series given in Appendix A. We obtain

$$\tfrac{1}{2}A_0 = (1 - e^2)^{-1/2}\left[1 + \sum_{j=1} 2c_j J_j(je)\right] \quad (7)$$

$$A_j = 2(1 - e^2)^{-1/2}\left\{c_j + \sum_{k=1} J_k(ke)[c_{|j-k|} + c_{j+k}]\right\} \quad (8)$$

$$B_j = (2/e)(1 - e^2)^{1/2}\sum_{k=1} kJ_k(ke)\{J_{|k-j|}(|k - j|e) - J_{k+j}[(k + j)e]\} \quad (9)$$

where c_j is defined by Eq. (A4).

Method 2: Hansen-Hill's Expansion

The coefficients of the Fourier expansion can be written in terms of Bessel functions and hypergeometric functions by means of Hansen-Hill's expansion as shown in Appendix B.

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After some manipulation the results can be expressed as follows:

$$\tfrac{1}{2}A_0 = (1+\beta^2)^2 G_0 \tag{10}$$

$$A_j = 2(1+\beta^2)^2\Big\{J_j(je)G_0 + \sum_{k=1}[J_{j-k}(je) + J_{j+k}(je)]G_k\Big\} \tag{11}$$

$$B_j = (1+\beta^2)^2\Big\{H_0\Big[\sum_{k=0}\beta^k J_{j-1+k}(je) - \sum_{k=1}^{j+1}\beta^k J_{j+1-k}(je)\Big] + \sum_{k=1}^{j-1} H_k J_{j-1-k}(je) - \sum_{k=0} H_k J_{j+1+k}(je)\Big\} \tag{12}$$

where

$$G_k = (1+k)\beta^k F(k+2,2,k+1,\beta^2) \quad k = 0,1,2,\ldots \tag{13}$$

$$H_k = \beta^k\binom{3}{k}F(k+3,1,k+1,\beta^2) \quad k = 0,1,2,\ldots \tag{14}$$

and $\binom{3}{k}$ is the binomial coefficient. The hypergeometric function $F(a,b,c,\beta^2)$ and β are defined in Appendices A and B, respectively.

The values of A_0–A_6 and B_1–B_6 are plotted vs the eccentricity as shown in Figs. 1 and 2, respectively. It is noted that the rate of convergence of both series is greatly reduced as the eccentricity increases.

In the following, two types of plane librational motion of a satellite, namely, small and finite oscillations, are considered.

Small oscillation

For small oscillations we may use the approximation $\sin 2\Psi = 2\Psi$. By making use of Eqs. (5) and (6), Eq. (4) can be written in the form of the well-known Hill's equation with a forcing function on the rhs:

$$d^2\Psi/dz^2 + \Big(\theta_0 + \sum_{r=1} 2\theta_{2r}\cos 2rz\Big)\Psi = \sum_{r=1} 8eB_r \sin 2rz \tag{15}$$

where $z = \tfrac{1}{2}M$ and $\theta_{2r} = 3KA_r$. We may use the solutions,[11] $\Psi_1(z) = ce_v(z,-\theta_2)$ and $\Psi_2(z) = se_v(z,-\theta_2)$, of the Mathieu equation

$$d^2\Psi/dz^2 + (\theta_0 + 2\theta_2\cos 2z)\Psi = 0$$

as a first approximation for $\Psi(z)$, i.e.,

$$\Psi(z) = A\Psi_1(z) + B\Psi_2(z)$$

The functions $ce_v(z,-\theta_2)$ and $se_v(z,-\theta_2)$ are the Mathieu functions of order v. The value of v is determined from

$$\theta_0 = v^2 + \frac{1}{2(v^2-1)}\theta_2^2 + \frac{5v^2+7}{32(v^2-1)^3(v^2-4)}\theta_2^4 + \ldots \tag{16}$$

Then, for the second approximation, $\Psi(z)$ is the solution of

$$d^2\Psi/dz^2 + (\theta_0 + 2\theta_2\cos 2z)\Psi = f(z) \tag{17}$$

where

$$f(z) = -\Big(\sum_{r=2} 2\theta_{2r}\cos 2rz\Big)ce_v(z,-\theta_2) + \sum_{r=1} 8eB_r \sin 2rz$$

The complete solution of Eq. (17) is

$$\Psi = A\Psi_1(z) + B\Psi_2(z) - \frac{1}{c^2}\Big[\Psi_1(z)\int_0^z \Psi_2(u)f(u)du - \Psi_2(z)\int_0^z \Psi_1(u)f(u)du\Big] \tag{18}$$

where the arbitrary constants A and B can be determined from the initial conditions and $c^2 = \Psi_1(0)(d\Psi_2/dz)_0 - (d\Psi_1/dz)_0\Psi_2(0)$. The detailed treatment of Mathieu functions and Hill's equation can be found elsewhere.[11]

Finite oscillation

If the magnitude of librational motion of a satellite is not sufficiently small for the linearization of the sine function, let us assume the solution of the periodic motion of Ψ with zero initial value is given by

$$\Psi = \sum_{i=1} a_i\alpha^i \sin iM \tag{19}$$

where α is a small parameter and the a_i's are to be determined. Based on Lagrange's theorem of expansion for the sine function of a series[9]

$$\sin 2\Psi = \sin\Big(\sum_{j=1} 2a_i\alpha^i \sin iM\Big) = \sum_{j=1} 2g_j \sin jM \tag{20}$$

where the formulation of the g_j's as functions of the a_i's is given in Appendix C. Now, combining the series given by Eqs. (5) and (20), we obtain

$$(a/r)^3 \sin 2\Psi = \sum_{n=1} c_n \sin nM \tag{21}$$

where

$$c_n = \sum_{k=1} g_k[A_{|n-k|} - A_{n+k}]$$

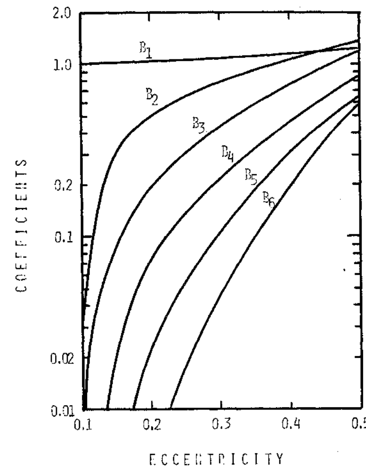

Fig. 2 The coefficients of expansion of $(a/r)^3$ sinf vs e.

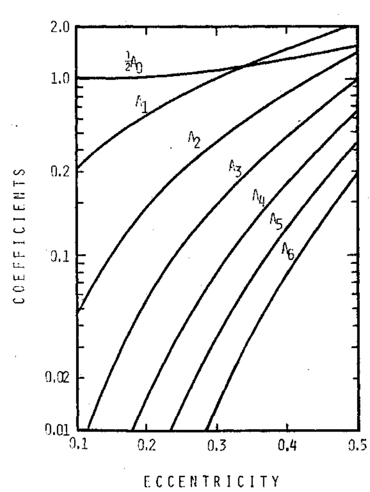

Fig. 1 The coefficients of expansion of $(a/r)^3$ vs e.

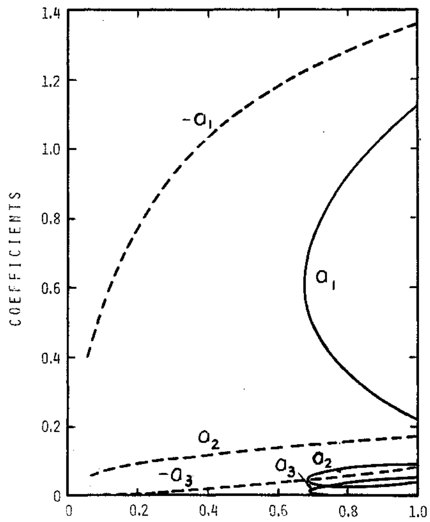

Fig. 3 The coefficients a_i of series solution vs K for e = 0.2.

Substitution of Eqs. (6) and (21) into Eq. (4) yields

$$d^2\Psi/dM^2 + \tfrac{3}{2}K \sum_{n=1} c_n \sin nM = \sum_{n=1} 2eB_n \sin nM \quad (22)$$

By substituting Eq. (19) into Eq. (22) and equating coefficients of $\sin nM$ of the resulting equation we obtain a set of nonlinear algebraic equations for the a's

$$-n^2\alpha^n a_n + \tfrac{3}{2}KC_n = 2eB_n \qquad n = 1,2, \ldots \quad (23)$$

If only the first seven terms of Eq. (19) are taken and α is put to unity, Eq. (23) reduces to

$$-2i^2a_i + 3K \sum_{j=1}^{7} c_{|i-j|,i+j}g_j = 4eB_i \qquad i = 1,2, \ldots, 7 \quad (24)$$

where $c_{ij} = A_i - A_j$ there seems no convenient numerical method for the solution of these highly nonlinear algebraic equations. However, it is plausible to obtain an approximate solution by considering that a_1 and a_2 are much greater than the other members in the set and hence, neglecting all the terms containing a_3–a_7 in the g functions. Consequently, the first two equations of Eq. (24) become two cubic equations of a_2

$$\alpha_i a_2^3 + \beta_i a_2^2 + \gamma_i a_2 + \delta_i = 0 \qquad i = 1,2 \quad (25)$$

where the coefficients are functions of a_1. Then, the solution of Eq. (25) can be obtained graphically by plotting a_2 as functions of a_1. In the neighborhood of the intersection of the two curves, a_2 of the first equation has three real roots and a_2 of the second has only one real root, and hence, the solution is unique. Calculation of the remaining a's is straightforward. Evaluating the g functions with the first approximation of a_1 and a_2, we may determine a_3–a_7 independently by using the third through seventh equations of Eq. (24), respectively. To obtain the second approximation of a_1 and a_2 we simply substitute the first approximation of a_3–a_7 as constants in the g functions and calculate the improved coefficients of Eq. (25) and repeat the process. The first approximate solutions of the a's for the case of $e = 0.2$ with the parameter $K = 0.1$–$K = 1.0$ have been obtained and plotted in Fig. 3. The results indicate that the assumption of a_3–a_7 much smaller that a_1 and a_2 is acceptable. In the region where K is greater than 0.66, there are three sets of solutions for the a's with two positive values and one negative value for a_1. The dashed curves represent the negative a's.

Solution of Ψ in terms of f

We may apply a similar approach for the solution by Eq. (3) and let

$$\Psi = \sum_{m=1}^{7} b_m \sin mf \quad (26)$$

$$\sin 2\Psi = \sum_{i=1}^{7} 2g_i \sin if \quad (27)$$

where the g's are functions of the b's as defined by Eq. (C3). Substituting Eqs. (26) and (27) into Eq. (3) and equating the coefficients of the equal harmonics, we obtain

$$\begin{aligned} -b_1 + 3Kg_1 &= 2e \\ -3eb_1 - 8b_2 - 3eb_3 + 6Kg_2 &= 0 \\ -4eb_2 - 9b_3 - 4eb_4 + 3Kg_3 &= 0 \\ \text{etc.} \end{aligned} \quad (28)$$

For the first approximation we neglect the terms containing b_3 and b_7 in the first two equations of Eq. (28) and obtain

$$\begin{aligned} (b_2^2)_1 &= [f_1(b_1) - (2e + b_1)/3K]/f_2(b_1) \\ (b_2)_2 &= 3eb_1/\{6K[f_3(b_1) - f_4(b_1)(b_2^2)_1] - 8\} \end{aligned} \quad (29)$$

where the functions $f_i(b_1)$ can be readily obtained by using Eqs. (28) and (C3). It is noted that an additional term $-1/144(1 - b_1^2/20)b_1^7$ should be added to $f_1(b_1)$ to improve its accuracy. This term is obtained from the higher order approximation of g_1. Equation (29) can be solved graphically and then the first approximation of b_3–b_7 can be determined independently from the third through the seventh equation of Eq. (28). The numerical results are plotted in Fig. 4.

3. Concluding Remarks

1) The mean motion of the small oscillatory libration of a satellite belongs to the class of Hill's equation with a sinusoidal forcing function. The term mean motion is used, since the independent variable is the mean anomaly of the elliptic orbit.

2) The solution for the periodic finite oscillations has been presented in the form of a seven-term Fourier series. The numerical results shown in Figs. 3 and 4 are the first approximate solution for the system of nonlinear algebraic

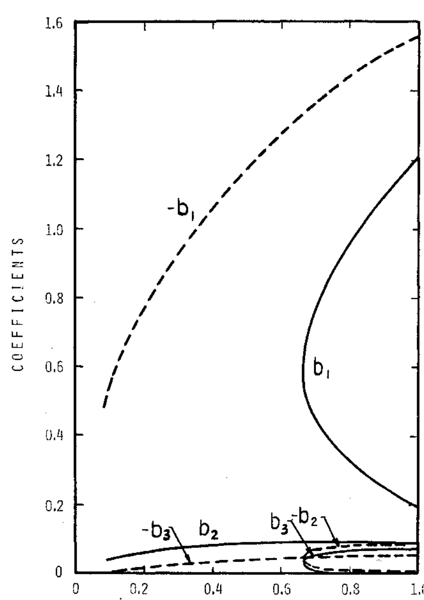

Fig. 4 The coefficients b_i of series solution vs K for e = 0.2.

equations given by Eqs. (24) and (28), respectively. In the region where K is approximately greater than 0.66, there are three sets of solutions for both the a's and b's. This result is in good agreement with that of Ref. 3. It can be seen that the series converges rapidly for small and moderate eccentricity of the orbit.

3) It is interesting to compare the two series solutions with the exact solution by a digital computer. The values $e = 0.2$ and $K = 0.9$ were chosen and three sets of solutions were obtained. The results are given in Table 1. The first and third sets of both series are compared with the exact periodic solutions as shown in Fig. 5 and it shows good agreement.

4) The initial dimensionless angular velocity of the series solutions are given by

$$(d\Psi/df)_0 = \sum_{k=1}^{3} kb_k = (1-e)^{3/2}(1+e)^{-1/2}\sum_{k=1}^{3} ka_k$$

and the values of $(d\Psi/df)_0$ of the exact periodic solution are obtained by using an iterative scheme. The results are shown in Table 2. Thus, the first approximation of b-series shows a closer agreement in both Ψ and $(d\Psi/df)_0$ with an exact solution.

Appendix A: Series Expansion

The following series are well known[8,9]:

$$a/r = 1 + \sum_{n=1}^{\infty} 2J_n(ne)\cos nM \tag{A1}$$

$$(a/r)^2 \sin f = \frac{1}{e}(1-e^2)^{1/2}\sum_{n=1}^{\infty} 2nJ_n(ne)\sin nM \tag{A2}$$

$$(a/r)^2 = (1-e^2)^{-1/2}\left[1 + \sum_{j=1}^{\infty} 2c_j \cos jM\right] \tag{A3}$$

where

$$c_j = \sum_{i=-\infty}^{\infty} J_i(je)\beta^{|i-j|}$$

$$\beta = [1-(1-e^2)^{1/2}]/e \tag{A4}$$

and $J_n(ne)$ is the Bessel function of the first kind of nth order.

Appendix B: Hansen-Hill's Expansion

As shown in Ref. 10, the coefficients of Fourier series expansion of the following:

$$(r/a)^n \sin mf = \sum_{i=1}^{\infty} B_i \sin iM$$

$$(r/a)^n \cos mf = \tfrac{1}{2}A_0 + \sum_{i=1}^{\infty} A_i \cos iM$$

can be expressed in the forms

$$A_0 = 2X_0^{n,m},\ A_i = X_i^{n,m} + X_{-i}^{n,m} \tag{B1}$$

$$B_i = X_i^{n,m} - X_{-i}^{n,m} \tag{B2}$$

Table 1 Coefficients of series solutions for $e = 0.2$ and $K = 0.9$

	a-series				b-series		
a_1	0.265	−0.005	−0.002	b_1	0.248	−0.051	0.0053
a_2	1.035	0.075	0.050	b_2	1.097	−0.086	0.047
a_3	−1.345	−0.160	−0.075	b_3	−1.454	0.089	−0.076

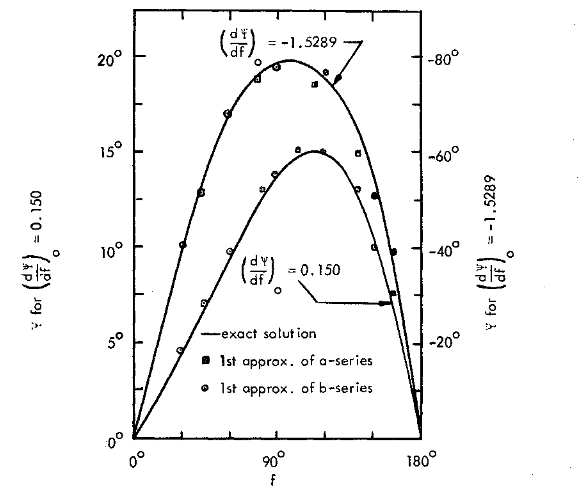

Fig. 5 Comparison of series solutions with the exact solution for $e = 0.2$ and $K = 0.9$.

where

$$X_i^{n,m} = (1+\beta^2)^{-n-1}\sum_{p=-\infty}^{\infty} J_p(ie)X_{i,p}^{n,m} \tag{B3}$$

The Hansen's coefficients $X_{i,p}^{n,m}$ are defined as

$i - p - m = 0,\ X_{i,p}^{n,m} = F(m-n-1, -m-n-1, \beta^2)$

$i - p - m > 0,\ X_{i,p}^{n,m} =$

$$\binom{n+1-m}{i-p-m}F(i-p-n-1, -m-n-1, i-p-m+1, \beta^2)(-\beta)^{i-p-m}$$

$i - p - m < 0,\ X_{i,p}^{n,m} =$

$$\binom{n+1+m}{-i+p+m}F(-i+p-n-1, m-n-1, -i+p+m+1, \beta^2)(-\beta)^{-i+p+m}$$

where the binomial coefficient

$$\binom{i}{p} = \frac{i(i-1)(i-2)\ldots(i-p+1)}{p!}$$

and the hypergeometric function

$$F(a,b,c,\beta^2) = 1 + \frac{a\cdot b}{1\cdot c}\beta^2 + \frac{a(a+1)b(b+1)}{1\cdot 2\, c(c+1)}\beta^4 + \ldots$$

in which a, b, and c are constants and β is defined in Appendix A.

Appendix C: Lagrange's Expansion of the Sine Function of a Series

The expansion of a sine function of a series can be obtained by applying the method developed in Ref. 9. Let us consider the seven term Fourier series

$$\sin\left(2\sum_{k=1}^{7} a_k \sin kM\right) = 2\sum_{j=1}^{7} g_j \sin jM \tag{C1}$$

Table 2 Comparison of initial dimensionless angular velocities $(d\Psi/df)_0$

a-series	b-series	Exact
0.163	0.152	0.150
−1.236	−1.504	−1.5289

and define the following:

$$A = \sum_{k=1}^{7} a_k z^k$$

$$A + A^3/3! + A^5/5! + \ldots = \sum_{k=1}^{\infty} p_n z^n$$

$$A^2/2! + A^4/4! + \ldots = \sum_{n=2}^{\infty} q_n z^n$$

$$r_0 = 1$$

$$r_1 = 0 \qquad s_1 = a_1$$

$$r_2 = \tfrac{1}{2}a_1^2 \qquad s_2 = a_2$$

$$r_3 = a_1 a_2 \qquad s_3 = a_3 + a_1^3/6$$

Then, the coefficient g_j of the series expansion given by Eq. (C1) is

$$g_j = \sum_{n=0}^{m} r_n p_{n+j}' - \sum_{n=1}^{m} s_n q_{n+j} \; j = 1,2, \ldots, 7 \qquad \text{(C2)}$$

where m is the largest integer equal or less than $\frac{1}{2}(7 - j)$. This gives

$$g_1 = a_1 - \tfrac{1}{2}a_1^3 + \tfrac{1}{2}a_1^2(a_3 + a_1^3/6) - a_2^2 a_1 + a_1 a_2(a_4 + \tfrac{1}{2}a_1^2 a_2) - (a_3 + a_1^3/6)(\tfrac{1}{2}a_2^2 + a_1 a_3 + a_1^4/24)$$

$$g_2 = a_2 - a_1^2 a_2 + \tfrac{1}{2}a_1^2(a_4 + \tfrac{1}{2}a_1^2 a_2) - a_2(\tfrac{1}{2}a_2^2 + a_1 a_3 + a_1^4/24) \qquad \text{(C3)}$$

$$g_3 = a_3 + a_1^3/6 - a_1(\tfrac{1}{2}a_2^2 + a_1 a_3 + a_1^4/24) + \tfrac{1}{2}a_1^2(a_5 + a_1 a_2^2/6 + \tfrac{1}{2}a_1^2 a_3 + a_1^5/120) - a_2(a_1 a_4 - a_2 a_3)$$

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Influence of Damping on the Dynamic Stability of Spherical Caps under Step Pressure Loading

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An explicit numerical procedure for solving the nonlinear axisymmetric shell equations of Sanders for dynamic loadings is presented. Dynamic buckling pressures are determined as a function of the geometric parameter λ for several values of damping.

Introduction

The transient response of thin shells to dynamic loading conditions is a subject of considerable practical interest. It has thus attracted the attention of numerous researchers in structural mechanics. Until quite recently, attention had largely been focused on linear problems, but the success of numerical procedures in the area of nonlinear static problems naturally encouraged an attack on the problem of dynamic buckling of thin shells. As a result, several interesting articles have recently appeared[1-4] which present reliable techniques and accurate results. These authors employ finite-difference procedures (Stricklin and Martinez[4] use a finite-element approach) to solve the nonlinear partial differential equations governing the axisymmetric large deflections of spherical caps. They also chose an implicit finite-difference scheme rather than an explicit technique. Because of the nonlinear character of the governing equations, the selection of an implicit procedure requires that, to advance the solution, at each time step one must solve a set of nonlinear algebraic equations whose order is equal to a multiple of the number of mesh points in space. Thus, an iterative procedure is required, and, as Archer and Lange[1] observe, unless one is chosen with at least a quadratic rate of convergence, the number of iterations required increases dramatically. Archer and Lange were the first to employ a completely numerical approach to the nonlinear shell problems. These authors

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